From last time:

Question: If $B = \{1,2,3\}$, what are the possible equivalence relations on B?

Solution: $(1,1), (2,2), (3,3)$ must be in any relation.

$R_1 = \{(1,1),(2,2),(3,3)\}$ ← the equals relation.

$R_2 = \{(1,1),(2,2),(3,3),(1,2),(2,1)\}$

$R_3 = \{(1,1),(2,2),(3,3),(2,3),(3,2)\}$.

$R_4 = \{(1,1),(2,2),(3,3),(1,3),(3,1)\}$

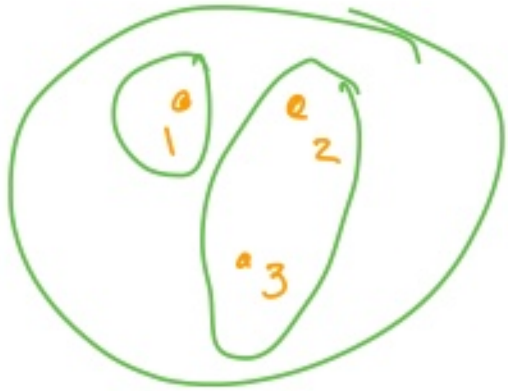
$R_5 = \{(1,1),(2,2),(3,3),(1,2),(2,1),(2,3),(3,2),(1,3),(3,1)\}$ ← everything equal

$= B \times B$

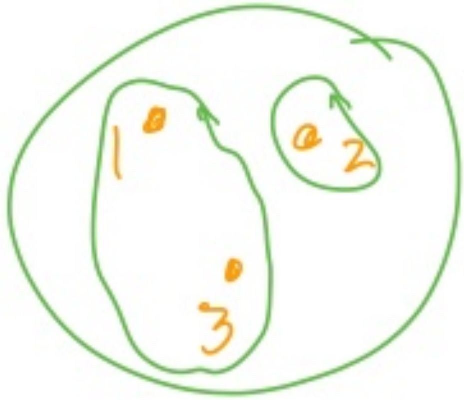
R_1 (equals)

R_2

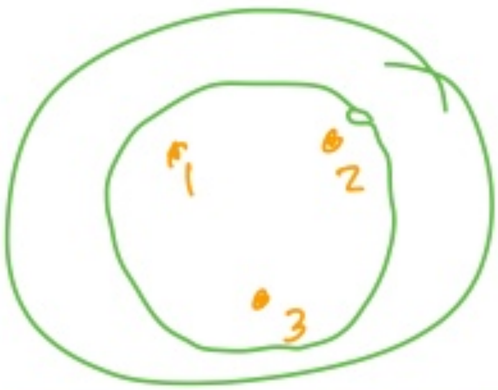
R_3



R_4



R_5



everything is equal.

An equivalence relation on a set S is actually a partition of the set into disjoint subsets $S = \bigcup S_j = \coprod S_j$, where all the things in the disjoint subsets S_j are equivalent to each other.

The subsets S_j are called <u>equivalence classes</u>.

Important example: An equivalence relation on $\mathbb{Z}$ is called "mod n", where n is a fixed integer. We say, if $a, b \in \mathbb{Z}$,

$$a \equiv b \bmod n \text{ if } (a-b) = kn$$

for some integer k.

Example Mod 3 equivalence on $\mathbb{Z}$.

$1 \equiv 4 \bmod 3$ because $1-4 = -3 = (-1)3$.

$1 \equiv 16 \bmod 3$ because $1-16 = -15 = (-5)3$.

The equivalence classes are

$$S_0 = \{0, 3, -3, 6, -6, 9, -9, \dots\} = 3\mathbb{Z}$$

$$S_1 = \{1, 4, -2, 7, -5, \dots\} = 1 + 3\mathbb{Z}.$$

$$S_2 = \{2, 5, -1, 8, -4, \dots\} = 2 + 3\mathbb{Z}.$$

$$\mathbb{Z} = S_0 \sqcup S_1 \sqcup S_2$$

$$(5+7) \equiv 5 \bmod 3 + 7 \bmod 3$$

$$\equiv 2 \bmod 3 + 1 \bmod 3 \equiv 3 \bmod 3 \equiv 0 \bmod 3$$

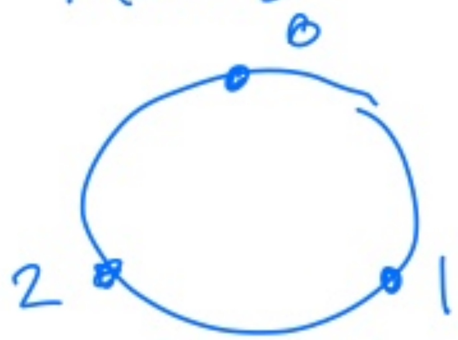
$$(5 \cdot 7) \equiv (2 \cdot 1) \bmod 3 \equiv 2 \bmod 3.$$

Example: Clock mod 12.

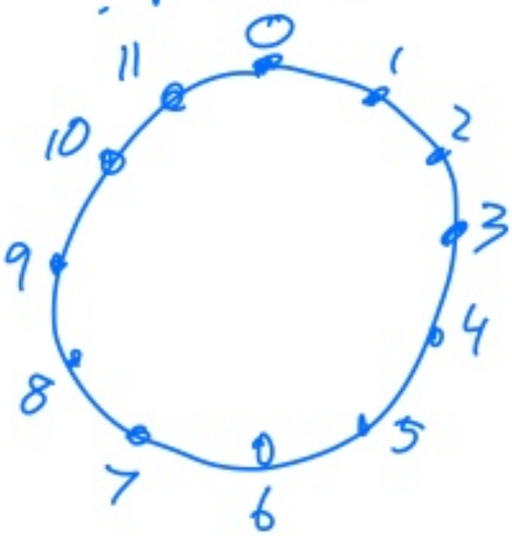
3 AM + 27 hours $\rightarrow$ 6 o'clock

$$(3+27) \bmod 12 \equiv (3+3) \bmod 12 = 6 \bmod 12.$$

Mod 3



Mod 12



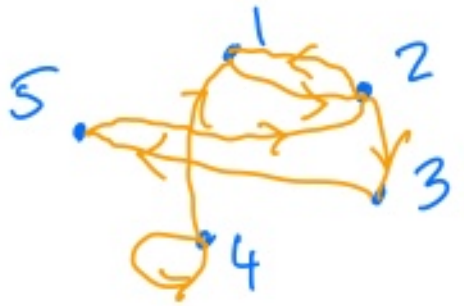
(Matrix Representations of Binary Relations
Graph Representations of Binary Relations.)

Set $S = \{1, 2, 3, 4, 5\}$

Relation $R = \{(1,2), (3,1), (3,5), (4,4), (4,1), (5,2), (2,3)\}$

Graphical representation

(vertex for each element of S). $(a,b) \in R \iff$ there is a directed edge from a to b.



Graph of the Relation R.

Matrix Representation : If $S = \{s_1, s_2, \dots, s_k\}$

$$R = \{(s_i, s_j) \dots \}$$

Rows & columns – correspond to the elements of S.

$$(s_i, s_j) \in R \iff \text{Matrix } M_{ij} = 1$$

$$\text{otherwise } M_{ij} = 0$$

example

$$S = \{1, 2, 3, 4, 5\}$$

$$R = \{(1,2), (3,1), (3,5), (4,4), (4,1), (5,2), (2,3)\}$$

matrix repⁿ of of the relation R

$$\Rightarrow M_R = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$